

On a new property of n -poised and GC_n sets

Vahagn Bayramyan, Hakop Hakopian

Abstract

In this paper we consider n -poised planar node sets, as well as more special ones, called GC_n -sets. For these sets all n -fundamental polynomials are products of n linear factors as it always takes place in the univariate case. A line ℓ is called k -node line for a node set \mathcal{X} if it passes through exactly k nodes. An $(n+1)$ -node line is called maximal line. In 1982 M. Gasca and J. I. Maeztu conjectured that every GC_n -set possesses necessarily a maximal line. Till now the conjecture is confirmed to be true for $n \leq 5$. It is well-known that any maximal line M of \mathcal{X} is used by each node in $\mathcal{X} \setminus M$, meaning that it is a factor of the fundamental polynomial of each node. In this paper we prove, in particular, that if the Gasca-Maeztu conjecture is true then any n -node line of GC_n -set \mathcal{X} is used either by exactly $\binom{n}{2}$ nodes or by exactly $\binom{n-1}{2}$ nodes. We prove also similar statements concerning n -node or $(n-1)$ -node lines in more general n -poised sets. This is a new phenomenon in n -poised and GC_n sets. At the end we present a conjecture concerning any k -node line.

Key words: Polynomial interpolation, Gasca-Maeztu conjecture, n -poised set, n -independent set, GC_n -set, fundamental polynomial, algebraic curve, maximal curve, maximal line.

Mathematics Subject Classification (2010):

primary: 41A05, 41A63; secondary 14H50.

1 Introduction and background

Let Π_n be the space of bivariate polynomials of total degree at most n :

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j \right\}.$$

We have that

$$N := \dim \Pi_n = \binom{n+2}{2}. \quad (1.1)$$

We say that a polynomial q is of degree k if $q \in \Pi_k \setminus \Pi_{k-1}$. Consider a set of s distinct nodes

$$\mathcal{X}_s = \{(x_1, y_1), (x_2, y_2), \dots, (x_s, y_s)\}.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, 2, \dots, s, \quad (1.2)$$

is called interpolation problem.

Let us now describe briefly the content of the paper. We consider here n -poised node sets for which the bivariate interpolation problem is unisolvant. We pay a special attention to a subclass of these sets called GC_n -sets. In such sets all n -fundamental polynomials, i.e., polynomials of total degree n vanishing at all nodes but one, are products of n linear factors. Note that this condition always takes place in the univariate case. A line ℓ is called k -node line for \mathcal{X} if it passes through exactly k nodes of \mathcal{X} . It is easily seen that at most $n+1$ nodes in an n -poised set (and therefore in a GC_n set) can be collinear. That is why $(n+1)$ -node line is called maximal line. In 1982 M. Gasca and J. I. Maeztu conjectured [12] that every GC_n -set possesses necessarily a maximal line. Till now the conjecture is confirmed to be true for $n \leq 5$ (see Subsection 1.3). We say that a node of an n -poised or GC_n -set uses a line if the line is a factor in the fundamental polynomial of the node. It is well-known that any maximal line M of \mathcal{X} is used by all nodes in $\mathcal{X} \setminus M$. Note that this statement, as well as the previous one concerning the maximal number of collinear nodes, follow readily from a well-known and simple fact that a bivariate polynomial of total degree at most n vanishes on a line if it vanishes at $n+1$ points in the line (see forthcoming Proposition 1.7). In Section 3 we prove that the subset of nodes of \mathcal{X} using a given k -node line is $(k-2)$ -independent, meaning that each node of the subset possesses a fundamental polynomial of total degree not exceeding $k-2$. In Sections 3.2 and 3.3 we prove a main result of this paper. Namely, if the Gasca-Maeztu conjecture is true then any n -node line of a GC_n -set \mathcal{X} is used either by exactly $\binom{n}{2}$ nodes or by exactly $\binom{n-1}{2}$ nodes. In Sections 2 and 2.2 similar statements are proved concerning n -node or $(n-1)$ -node lines in n -poised sets. Let us mention that this is a new phenomenon in n -poised and GC_n sets. At the end we present a conjecture concerning any k -node line.

Now let us go to exact definitions and formulations.

Definition 1.1. The set of nodes \mathcal{X}_s is called *n -poised* if for any data $\{c_1, \dots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (1.2).

A polynomial $p \in \Pi_n$ is called an *n -fundamental polynomial* for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik}, \quad i = 1, \dots, s,$$

where δ is the Kronecker symbol. We denote the n -fundamental polynomial of $A \in \mathcal{X}_s$ by $p_A^* = p_{A, \mathcal{X}_s}^*$. Sometimes we call fundamental also a polynomial that vanishes at all nodes but one, since it is a nonzero constant times the fundamental polynomial.

In view of the uniqueness we get readily that for any n -poised set the degree of each fundamental polynomial equals to n .

A necessary condition of n -poisedness of \mathcal{X}_s is: $|\mathcal{X}_s| = s = N$.

The following is a Linear Algebra fact:

Proposition 1.2. *The set of nodes \mathcal{X}_N is n -poised if and only if the following implication holds for any polynomial $p \in \Pi_n$:*

$$p(x_i, y_i) = 0, \quad i = 1, \dots, N \Rightarrow p = 0.$$

1.1 n -independent and n -dependent sets

Next we introduce an important concept of n -dependence of node sets:

Definition 1.3. A set of nodes \mathcal{X} is called *n -independent* if all its nodes have fundamental polynomials. Otherwise, \mathcal{X} is called *n -dependent*.

Clearly fundamental polynomials are linearly independent. Therefore a necessary condition of n -independence is $|\mathcal{X}| \leq N$.

Suppose a node set \mathcal{X}_s is n -independent. Then by using the Lagrange formula:

$$p = \sum_{A \in \mathcal{X}_s} c_A p_A^*,$$

we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.2).

Thus we get a simple characterization of n -independence:

A node set \mathcal{X}_s is n -independent if and only if the interpolation problem (1.2) is *n -solvable*, meaning that for any data $\{c_1, \dots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.2).

Now suppose that \mathcal{X}_s is n -dependent. Then some node (x_{i_0}, y_{i_0}) , does not possess an n -fundamental polynomial. This means that the following implication holds for any polynomial $p \in \Pi_n$:

$$p(x_i, y_i) = 0, \quad i \in \{1, \dots, s\} \setminus \{i_0\} \Rightarrow p(x_{i_0}, y_{i_0}) = 0.$$

In this paper we will deal frequently with a stronger version of n -dependence:

Definition 1.4. A set of nodes \mathcal{X} is called *essentially n -dependent* if none of its nodes possesses a fundamental polynomial.

Below, and frequently in the sequel, we use same the notation for a polynomial $q \in \Pi_k$ and the curve described by the equation $q(x, y) = 0$.

Remark 1.5. Suppose a set of nodes \mathcal{X} is essentially n -dependent and $q \in \Pi_k$, $k \leq n$, is a curve. Then we have that the subset $\mathcal{X}' := \mathcal{X} \setminus q$ is essentially $(n - k)$ -dependent, provided that $\mathcal{X}' \neq \emptyset$.

Indeed, suppose conversely that a node $A \in \mathcal{X}'$ has an $(n - k)$ -fundamental polynomial $r \in \Pi_{n-k}$. Then the polynomial $qr \in \Pi_n$ is an n fundamental polynomial of the node A in \mathcal{X} , which contradicts our assumption.

Definition 1.6. Given an n -poised set \mathcal{X} , we say that a node $A \in \mathcal{X}$ *uses* a curve $q \in \Pi_k$, if q divides the fundamental polynomial $p_{A,\mathcal{X}}^*$:

$$p_{A,\mathcal{X}}^* = qr, \quad \text{where } r \in \Pi_{n-k}.$$

The following proposition is well-known (see e.g. [14] Proposition 1.3):

Proposition 1.7. *Suppose that ℓ is a line. Then for any polynomial $p \in \Pi_n$ vanishing at $n + 1$ points of ℓ we have*

$$p = \ell r, \quad \text{where } r \in \Pi_{n-1}.$$

Evidently, this implies that any set of $n + 2$ collinear nodes is essentially n -dependent. We also obtain from Proposition 1.7

Corollary 1.8. *The following hold for any n -poised node set \mathcal{X} :*

- (i) *At most $n + 1$ nodes of \mathcal{X} can be collinear;*
- (ii) *A line ℓ containing $n + 1$ nodes of \mathcal{X} is used by all the nodes in $\mathcal{X} \setminus \ell$.*

In view of this a line ℓ containing $n + 1$ nodes of an n -poised set \mathcal{X} is called a maximal line (see [5]).

One can verify readily the following two properties of maximal lines of n -poised set \mathcal{X} :

- (i) Any two maximal lines of \mathcal{X} intersect necessarily at a node of \mathcal{X} ;
- (ii) Three maximal lines of \mathcal{X} cannot meet in one node.

Thus, in view of (1.1), there are no n -poised sets with more than $n + 2$ maximal lines.

1.2 Some results on n -independence

Let us start with the following simple but important result of Severi [24]:

Theorem 1.9 (Severi). *Any node set \mathcal{X} consisting of at most $n + 1$ nodes is n -independent.*

Next we consider node sets consisting of at most $2n + 1$ nodes:

Proposition 1.10 ([11]). *Any node set \mathcal{X} consisting of at most $2n+1$ nodes is n -dependent if and only if $n+2$ nodes are collinear.*

For a generalization of above two results for multiple nodes see [24] and [13], respectively.

The third result in this series is the following

Proposition 1.11. *Any node set \mathcal{X} consisting of at most $3n-1$ nodes is n -dependent if and only if at least one of the following holds.*

- (i) $n+2$ nodes are collinear,
- (ii) $2n+2$ nodes belong to a conic (possibly reducible).

Let us mention that this result as well as the two previous results are special cases of the following

Theorem 1.12 ([17], Thm. 5.1). *Any node set \mathcal{X} consisting of at most $3n$ nodes is n -dependent if and only if at least one of the following holds.*

- (i) $n+2$ nodes are collinear,
- (ii) $2n+2$ nodes belong to a conic,
- (iii) $|\mathcal{X}| = 3n$, and there is a cubic $\gamma \in \Pi_3$ and an algebraic curve $\sigma \in \Pi_n$ such that $\mathcal{X} = \gamma \cap \sigma$.

1.3 GC_n sets and the Gasca-Maeztu conjecture

Let us consider a special type of n -poised sets whose n -fundamental polynomials are products of n linear factors as it always takes place in the univariate case:

Definition 1.13 (Chung, Yao [10]). An n -poised set \mathcal{X} is called GC_n -set if the n -fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

In other words, GC_n sets are the sets each node of which uses exactly n lines.

Now we are in a position to present the Gasca-Maeztu conjecture, called briefly also GM conjecture:

Conjecture 1.14 (Gasca, Maeztu, [12]). *Any GC_n -set contains $n+1$ collinear nodes.*

Thus the GM conjecture states that any GC_n set possesses a maximal line. So far, this conjecture has been verified for the degrees $n \leq 5$. For $n = 2$, the conjecture is evidently true. The case $n = 3$ is not hard to prove. The case $n = 4$ was proved for the first time by J. R. Busch in 1990 [6]. Since then, four other proofs have appeared for this case: [8, 15, 2], and [25]. In

our opinion the last one is the simplest and the shortest one. The case $n = 5$ was proved recently in [16].

Notice that if a line M is maximal then the set $\mathcal{X} \setminus M$ is $(n - 1)$ -poised. Moreover, if \mathcal{X} is a GC_n -set then $\mathcal{X} \setminus M$ is a GC_{n-1} -set. For a generalization of the Gasca-Maeztu conjecture to maximal curves see [18].

In the sequel we will make use of the following important result of Carnicer and Gasca concerning the GM conjecture:

Theorem 1.15 (Carnicer, Gasca, [9]). *If the Gasca-Maeztu conjecture is true for all $k \leq n$, then any GC_n -set possesses at least three maximal lines.*

In view of this one gets readily that each node of \mathcal{X} uses at least one maximal line.

1.4 Some examples of n -poised and GC_n sets

We will consider 3 well-known constructions: *The Berzolari-Radon construction* [4, 22], *the Chung-Yao construction* [10] (called also *Chung-Yao natural lattice*), and the *principal lattice*. The first construction gives examples of n -poised sets, while the remaining two give examples of GC_n -sets. Let us mention that both the Chung-Yao natural lattice and the principal lattice are special cases of the Berzolari-Radon construction.

Note that Lagrange and Newton formulas for these constructions can be found in [12] and [21].

The Berzolari-Radon construction

A set \mathcal{X} containing $N = 1 + 2 + \dots + (n + 1)$ nodes is called Berzolari-Radon set if there are $n + 1$ lines: $\ell_1, \dots, \ell_{n+1}$ such that the sets $\ell_1, \ell_2 \setminus \ell_1, \ell_3 \setminus (\ell_1 \cup \ell_2), \dots, \ell_{n+1} \setminus (\bigcup_{i=1}^n \ell_i)$ contain exactly $(n + 1), n, (n - 1), \dots, 1$ nodes, respectively. The Berzolari-Radon set is n -poised.

It is worth noting that the Gasca-Maeztu conjecture is equivalent to the statement that every GC_n -set is a Berzolari-Radon set.

The Chung-Yao construction

Consider $n + 2$ lines: $\ell_1, \dots, \ell_{n+2}$, such that no two lines are parallel, and no three lines intersect in one point. Then the set \mathcal{X} of intersection points of these lines is called Chung-Yao set. Notice that $|\mathcal{X}| = \binom{n+2}{2}$. Each fixed node here is lying in exactly 2 lines, and does not belong to the remaining n lines. Moreover, the product of these n lines gives the fundamental polynomial of the fixed node. Thus \mathcal{X} is GC_n -set.

Note that this construction can be characterized by the fact that all the given $n + 2$ lines are maximal. As it was mentioned earlier, there are no n -poised sets with more maximal lines.

The principal lattice

The principal lattice is the following set (or an affine image of it)

$$\mathcal{X} = \{(i, j) \in \mathbb{Z}_+^2 : i + j \leq n\}.$$

Notice that the fundamental polynomial of the node (i, j) here uses i vertical lines: $x = k$, $k = 0, \dots, i - 1$, j horizontal lines: $y = k$, $k = 0, \dots, j - 1$ and $n - i - j$ lines with slope -1 : $x + y = k$, $k = i + j + 1, \dots, n$. Thus \mathcal{X} is GC_n -set.

Note that this lattice possesses just three maximal lines, namely the lines $x = 0$, $y = 0$, and $x + y = n$. Note that, according to Theorem 1.15, there are no n -poised sets with less maximal lines, provided that the Gasca-Maeztu conjecture is true.

1.5 Maximal curves and the sets \mathcal{N}_q and \mathcal{X}_ℓ

Let us start with a generalization of Proposition 1.7 for algebraic curves of higher degree. First set for $k \leq n$

$$d(n, k) := \dim \Pi_n - \dim \Pi_{n-k} = \frac{1}{2} k (2n + 3 - k).$$

Proposition 1.16 (Rafayelyan, [23], Prop. 3.1). *Let q be an algebraic curve of degree $k \leq n$ without multiple components. Then the following hold.*

- (i) *Any subset of q consisting of more than $d(n, k)$ nodes is n -dependent.*
- (ii) *A subset $\mathcal{X} \subset q$ consisting of $d(n, k)$ nodes is n -independent if and only if the following implication holds for any polynomial $p \in \Pi_n$:*

$$p|_{\mathcal{X}} = 0 \implies p = qr \quad \text{for some } r \in \Pi_{n-k}. \quad (1.3)$$

Let us mention that a special case of (i), when q factors into linear factors, is due to Carnicer and Gasca, [7].

As in the case of lines (see Corollary 1.8) we get readily from here

Corollary 1.17. *The following hold for any n -poised node set \mathcal{X} :*

- (i) *At most $d(n, k)$ nodes of \mathcal{X} can lie in a curve of degree k ;*
- (ii) *A curve of degree $k \leq n$ without multiple components containing $d(n, k)$ nodes of \mathcal{X} is used by all the nodes in $\mathcal{X} \setminus q$.*

Next we bring a generalization of the concept of a maximal line (see [23]):

Definition 1.18. A curve of degree $k \leq n$ without multiple components passing through $d(n, k)$ nodes of an n -poised set \mathcal{X} is called a *maximal curve* for \mathcal{X} .

Thus maximal line, conic, and cubic pass through $n + 1$, $2n + 1$, and $3n$ nodes of \mathcal{X} , respectively.

Below, for an n -poised set \mathcal{X} , line ℓ and an algebraic curve q , we define important sets \mathcal{X}_ℓ and \mathcal{N}_q , which will be used frequently in the sequel.

Definition 1.19. Let \mathcal{X} be an n -poised set ℓ be a line and q be an algebraic curve without multiple factors. Then

- (i) \mathcal{X}_ℓ is the subset of nodes of \mathcal{X} which use the line ℓ ;
- (ii) \mathcal{N}_q is the subset of nodes of \mathcal{X} which do not use the curve q and are not lying in q .

Next let us bring a characterization of maximal curves:

Proposition 1.20 (Rafayelyan, [23], Prop. 3.3). *Let \mathcal{X} be an n -poised set and q be an algebraic curve of degree $k \leq n$ without multiple factors. Then the following statements are equivalent:*

- (i) *The curve q is maximal for \mathcal{X} ;*
- (ii) *All the nodes in $\mathcal{Y} := \mathcal{X} \setminus q$ use the curve q , i.e., $\mathcal{N}_q = \emptyset$;*
- (iii) *The set \mathcal{Y} is $(n - k)$ -poised. Moreover, if \mathcal{X} is a GC_n -set then \mathcal{Y} is a GC_{n-k} -set.*

Thus $\mathcal{N}_q = \emptyset$ means that q is a maximal curve. The following result concerns the case when $\mathcal{N}_q \neq \emptyset$.

Proposition 1.21 (Rafayelyan, [23]). *Let \mathcal{X} be an n -poised set and q be an algebraic curve of degree $k \leq n$ without multiple factors. Then the set \mathcal{N}_q is essentially $(n - k)$ -dependent, provided that it is not empty.*

It is worth mentioning that the special case $k = 1$ of above two results, where q is a line is due to Carnicer and Gasca [8]. Note also that the case when q is a product of k lines is proved in [15].

Proposition 1.22 ([3]). *Let \mathcal{X} be an n -poised set. Then there is at most one algebraic curve of degree $n - 1$ passing through $N - 4$ nodes of \mathcal{X} .*

From here one gets readily for any n -poised set \mathcal{X} (see [1]):

$$|\mathcal{X}_\ell| \leq 1 \quad \text{if } \ell \text{ is a 2-node line.} \quad (1.4)$$

For a generalization of these results for curves of arbitrary degree and 3-node lines see [19, 20]. Let us mention that the statement (1.4) for GC_n sets has already been shown in [9].

In the sequel we will use frequently the following 2 lemmas from [9]. Let us mention that the the second lemma is used in a proof there and is not explicitly formulated. For the sake of completeness we bring proofs here.

Lemma 1.23 (Carnicer, Gasca, [9]). *Let \mathcal{X} be an n -poised set and ℓ be a line. Suppose also that there is a maximal line M_0 such that $M_0 \cap \ell \notin \mathcal{X}$. Then we have that*

$$\mathcal{X}_\ell = (\mathcal{X} \setminus M_0)_\ell.$$

If in addition ℓ is an n -node line then we have that

$$\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup M_0) \text{ and therefore } |\mathcal{X}_\ell| = \binom{n}{2}.$$

Proof. Suppose conversely that a node $A \in M_0$ uses ℓ :

$$p_A^\star = \ell q, \quad q \in \Pi_{n-1}.$$

Notice that q vanishes at the n nodes in $M_0 \setminus \{A\}$. Thus, in view of Proposition 1.7, we have that q and hence p_A^\star vanishes on M_0 . In particular p vanishes at A , which is a contradiction.

Now assume that ℓ is an n -node line and $A \notin \ell \cup M_0$. Then we have that

$$p_A^\star = M_0 q, \quad q \in \Pi_{n-1}.$$

Notice that q vanishes at the n nodes of ℓ . Thus, in view of Proposition 1.7, we have that $q = \ell r$, $r \in \Pi_{n-2}$. Thus we obtain that $p_A^\star = M_0 \ell r$, i.e., A uses the line ℓ . \square

Lemma 1.24 (Carnicer, Gasca, [9]). *Let \mathcal{X} be an n -poised set and ℓ be a line. Suppose also that there are two maximal lines M', M'' such that $M' \cap M'' \cap \ell \in \mathcal{X}$. Then we have that*

$$\mathcal{X}_\ell = (\mathcal{X} \setminus (M' \cup M''))_\ell.$$

If in addition ℓ is an n -node line then we have that

$$\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup M' \cup M'') \text{ and therefore } |\mathcal{X}_\ell| = \binom{n-1}{2}.$$

Proof. Suppose that a node $A \in (M' \cup M'') \setminus \ell$ uses ℓ :

$$p_A^\star = \ell q, \quad q \in \Pi_{n-1}.$$

Suppose, for example, $A \in M'$. Then notice that q vanishes at the n nodes of $M'' \setminus \ell$. Thus, in view of Proposition 1.7, we have that $q = M'' r$, $r \in \Pi_{n-2}$. Then r vanishes on $n-1$ nodes in $M' \setminus \ell$ different from A . Thus r and hence also p_A^\star vanishes on whole line M' including A , which is a contradiction.

Now assume that ℓ is an n -node line and $A \notin \ell \cup M' \cup M''$. Then we have that

$$p_A^\star = M' M'' q, \quad q \in \Pi_{n-2}.$$

Notice that q vanishes at the $n-1$ nodes of ℓ different from the node of intersection with the maximal lines. Thus, in view of Proposition 1.7, we have that $q = \ell r$, $r \in \Pi_{n-3}$. Therefore we obtain that $p_A^\star = M' M'' \ell r$, i.e., A uses the line ℓ . \square

2 Lines in n -poised sets

2.1 On n -node lines in n -poised sets

Let us start our results with the following

Proposition 2.1. *Let \mathcal{X} be an n -poised set and ℓ be a line passing through exactly n nodes of \mathcal{X} . Then the following hold:*

- (i) $|\mathcal{X}_\ell| \leq \binom{n}{2}$;
- (ii) *If $|\mathcal{X}_\ell| \geq \binom{n-1}{2} + 1$ then there is a maximal line M_0 such that $M_0 \cap \ell \notin \mathcal{X}$. Moreover, we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (M_0 \cup \ell)$. Hence it is an $(n-2)$ -poised set. In particular we have that $|\mathcal{X}_\ell| = \binom{n}{2}$;*
- (iii) *If $\binom{n-1}{2} \geq |\mathcal{X}_\ell| \geq \binom{n-2}{2} + 2$, then $|\mathcal{X}_\ell| = \binom{n-1}{2}$. Moreover, \mathcal{X}_ℓ is an $(n-3)$ -poised set and there is a conic $\beta \in \Pi_2$ such that $\mathcal{X}_\ell = \mathcal{X} \setminus (\beta \cup \ell)$. Furthermore, we have that $\mathcal{N}_\ell \subset \beta$ and $|(\beta \setminus \ell) \cap \mathcal{X}| = |\mathcal{N}_\ell| = 2n$. Besides these $2n$ nodes the conic may contain at most one extra node, which necessarily belongs to ℓ . If β is reducible: $\beta = \ell_1 \ell_2$ then we have that $|\ell_i \cap (\mathcal{X} \setminus \ell)| = n$, $i = 1, 2$.*

Proof. (i) Assume by way of contradiction that $|\mathcal{X}_\ell| \geq \binom{n}{2} + 1$. Then we obtain

$$|\mathcal{N}_\ell| \leq \binom{n+2}{2} - \left[\binom{n}{2} + 1 \right] - n = n.$$

This is a contradiction, since on one hand, in view of Proposition 1.20, the nonempty set \mathcal{N}_ℓ is $(n-1)$ -dependent and on the other hand, in view of Theorem 1.9, it is $(n-1)$ -independent.

(ii) In this case we have that

$$|\mathcal{N}_\ell| \leq \binom{n+2}{2} - \left[\binom{n-1}{2} + 1 \right] - n = 2n - 1 = 2(n-1) + 1.$$

Now let us make use of Proposition 1.10. Since \mathcal{N}_ℓ is $(n-1)$ -dependent, we get that there is a line M_0 passing through $n+1$ nodes of \mathcal{N}_ℓ . The line M_0 is maximal and therefore cannot pass through any more nodes. Hence we obtain that $M_0 \cap \ell \notin \mathcal{X}$. Thus, in view of Lemma 1.23, we have that \mathcal{X}_ℓ is an $(n-2)$ -poised set.

(iii) In this case we have that

$$|\mathcal{N}_\ell| \leq \binom{n+2}{2} - \left[\binom{n-2}{2} + 2 \right] - n = 3n - 4 = 3(n-1) - 1.$$

Since the set \mathcal{N}_ℓ is $(n-1)$ -dependent, we get from Proposition 1.11, that either

- a) there is a line M_0 passing through $n + 1$ nodes in \mathcal{N}_ℓ , or
- b) there is a conic $\beta \in \Pi_2$ passing through $2n = 2(n - 1) + 2$ nodes in \mathcal{N}_ℓ .

Let us start with the case a). We have for the maximal line M_0 , in the same way as in the case ii), that $M_0 \cap \ell \notin \mathcal{X}$ and therefore $|\mathcal{X}| = \binom{n}{2}$. This contradicts our assumption in iii).

In the case b) let us first show that $|\mathcal{N}_\ell| = 2n$. Indeed, in view of Proposition 1.20, we have that \mathcal{N}_ℓ is essentially $(n - 1)$ -dependent. Then, suppose \mathcal{N}_ℓ , besides the nodes in β , contains t nodes outside of it, where $t \leq n - 4 (= 3n - 4 - 2n)$. In view of Remark 1.5 these t nodes must be $(n - 3)$ -essentially dependent. Therefore, we get from Theorem 1.9 that $t = 0$. Now notice that $\ell\beta$ is a maximal cubic since it passes through $3n$ nodes. The conic β , besides the $2n$ nodes, may contain at most 1 extra node, since the set \mathcal{X} is n -independent. But, if the extra node does not belong to ℓ , then the cubic $\ell\beta$ would contain $3n + 1$ nodes, which is a contradiction.

Finally assume that the conic is reducible: $\beta = \ell_1\ell_2$. Then, since \mathcal{N}_ℓ is $(n - 1)$ -essentially dependent, we readily get that each of the lines passes through exactly n nodes from the $2n$. \square

2.2 On $(n - 1)$ -node lines in n -poised sets

Proposition 2.2. *Let \mathcal{X} be an n -poised set and ℓ be a line passing through exactly $n - 1$ nodes of \mathcal{X} . Assume also that $|\mathcal{X}_\ell| \geq \binom{n-2}{2} + 3$. Then we have that \mathcal{X}_ℓ is $(n - 3)$ -poised set. Hence $|\mathcal{X}_\ell| = \binom{n-1}{2}$ and $|\mathcal{N}_\ell| = 2n + 1$. Moreover, these $2n + 1$ nodes are located in the following way:*

- (i) $n + 1$ nodes are in a maximal line M_0 and
- (ii) n nodes are in an n -node line M'_0 .

Furthermore, besides these n nodes, the line M'_0 may contain at most one extra node, which necessarily belongs to M_0 .

Proof. We have that

$$|\mathcal{N}_\ell| \leq \binom{n+2}{2} - \left[\binom{n-2}{2} + 3 \right] - (n - 1) = 3n - 4 = 3(n - 1) - 1.$$

According to Proposition 1.20 the set \mathcal{N}_ℓ is essentially $(n - 1)$ -dependent. Therefore, in view of Proposition 1.11, we have that either

- (i) there is a line M_0 passing through $n + 1$ nodes of \mathcal{N}_ℓ , or
- (ii) there is a conic $\beta \in \Pi_2$ passing through $2n = 2(n - 1) + 2$ nodes of \mathcal{N}_ℓ .

Assume that i) holds. Then, suppose there are s nodes in \mathcal{N}_ℓ outside the line M_0 , where $s \leq 2(n - 2) - 1 (= 2n - 5 = 3n - 4 - n - 1)$.

Let us verify that $s \neq 0$. Assume conversely that $s = 0$. Then we have that any node $A \in \mathcal{X} \setminus (\ell \cup M_0)$ uses the line ℓ and maximal line M_0 , i.e.,

$$p_A^* = \ell M_0 q, \quad q \in \Pi_{n-2}.$$

This means, in view of Proposition 1.20 (part i) \Leftrightarrow ii)), that the conic ℓM_0 is maximal, which is contradiction, since it passes through only $2n$ nodes (instead of $2n + 1$ nodes).

Then, in view of Remark 1.5 these s nodes must be $(n-2)$ -essentially dependent. Therefore, by Proposition 1.10, there is a line M'_0 passing through n nodes of $\mathcal{N}_\ell \setminus M_0$. Now, suppose there are t nodes in \mathcal{N}_ℓ outside the lines M_0 and M'_0 , where $t \leq (n-3) - 2 (= n-5 = 2n-5-n)$. These t nodes, in view of Remark 1.5, must be essentially $(n-3)$ -dependent. Thus, we conclude from Theorem 1.9 that $t = 0$ and therefore $|\mathcal{N}_\ell| = 2n + 1$.

Now, it remains to verify that the case ii) is impossible.

Thus assume that ii) holds. Denote the number of nodes in \mathcal{N}_ℓ outside the conic β by t . We have that $t \leq (n-2) - 2 (= n-4 = 3n-4-2n)$. In view of Remark 1.5 these t nodes must be $(n-3)$ -essentially dependent. Therefore, by Theorem 1.9, we obtain that $t = 0$ and therefore $|\mathcal{N}_\ell| = 2n$.

Now we have that any node $A \in \mathcal{X} \setminus (\ell \cup \beta)$ uses the line ℓ . This means

$$p_A^* = \ell q, \quad q \in \Pi_{n-1}.$$

The curve q passes through all the $2n$ nodes in β . By the Bezout theorem we conclude that q divides β . Indeed, this is evident when β is irreducible. Now assume that β is reducible, i.e., $\beta = \ell_1 \ell_2$. The set \mathcal{N}_β is $(n-1)$ -essentially dependent. Therefore each line $\ell_i, i = 1, 2$, passes through exactly n nodes of \mathcal{N}_β and hence divides q . Thus we have that $q = \beta r$, $q \in \Pi_{n-3}$. Finally we get

$$p_A^* = \ell \beta r, \quad r \in \Pi_{n-3} \quad \text{for any } A \in \mathcal{X} \setminus (\ell \cup \beta).$$

This means that each node outside ℓ and β uses the reducible cubic $\ell \beta$. Therefore, by Proposition 1.20 (part i) \Leftrightarrow ii)), the latter curve is maximal, which is contradiction, since it passes through only $3n - 1$ nodes (instead of $3n$ nodes). \square

Corollary 2.3. *Let \mathcal{X} be an n -poised set and ℓ be a line passing through exactly $n - 1$ nodes of \mathcal{X} . Then we have that $|\mathcal{X}_\ell| \leq \binom{n-1}{2}$.*

Proof. Assume by way of contradiction that $|\mathcal{X}_\ell| \geq \binom{n-1}{2} + 1$. Notice that

$$\binom{n-1}{2} + 1 \geq \binom{n-2}{2} + 3 \text{ if } n \geq 4.$$

Now, in view of Proposition 2.2, we get that $|\mathcal{X}_\ell| \leq \binom{n-1}{2}$, which contradicts our assumption. It remains to note that Corollary in the case $n = 3$ is a special case in (1.4). \square

3 Lines in GC_n sets

3.1 On k -node lines in GC_n sets

Proposition 3.1. *Assume that Conjecture 1.14 holds for all degrees up to ν . Let \mathcal{X} be a GC_n set, $n \leq \nu$, and ℓ be a line passing through exactly k nodes of \mathcal{X} . Then the set \mathcal{X}_ℓ is $(k-2)$ -independent set. Moreover, for each node $A \in \mathcal{X}_\ell$ there is a $(k-2)$ -fundamental polynomial that divides the n -fundamental polynomial of A in \mathcal{X} .*

Proof. First suppose that $k = n+1$, meaning that ℓ is a maximal line. Then we have that $\mathcal{X}_\ell = \mathcal{X} \setminus \ell$ and this set is GC_{n-1} -set and hence is $(n-1)$ -poised.

In the case when ℓ is not maximal we will use induction on n . The case $n = 2$ is evident (see Subsection 3.3.2). Suppose Proposition is true for all degrees less than n and let us prove it for n .

Suppose that there is a maximal line M_0 such that $M_0 \cap \ell \notin \mathcal{X}$. Then we get from Lemma 1.23 that $\mathcal{X}_\ell = (\mathcal{X}_0)_\ell$ where $\mathcal{X}_0 := \mathcal{X} \setminus M_0$. We have that the set \mathcal{X}_0 is GC_{n-1} -set and ℓ passes through exactly k nodes of \mathcal{X}_0 . Therefore by induction hypothesis for the degree $n-1$ we get that \mathcal{X}_ℓ is $(k-2)$ -independent.

Now, in view of Theorem 1.15, consider three maximal lines for \mathcal{X} and denote them by M_i , $i = 1, 2, 3$. It remains to consider the case when each of these maximal lines intersects ℓ at a node of \mathcal{X} .

We will prove that \mathcal{X}_ℓ is $(k-2)$ -independent by finding a $(k-2)$ -fundamental polynomial for each node $A \in \mathcal{X}_\ell$. Since 3 maximal lines intersect each other at 3 distinct nodes there is $i_0 \in \{1, 2, 3\}$ such that $A \notin M_{i_0}$. We have that the set $\mathcal{Y} := \mathcal{X} \setminus M_{i_0}$ is GC_{n-1} -set and ℓ passes through exactly $k-1$ nodes of \mathcal{Y} . Therefore by induction hypothesis for the degree $n-1$ we get that the set \mathcal{Y}_ℓ is $(k-3)$ -independent. Moreover, there is a $(k-3)$ -fundamental polynomial $p_{A, \mathcal{Y}_\ell}^* \in \Pi_{k-3}$ which divides $p_{A, \mathcal{Y}}^*$.

Now, since $\mathcal{X}_\ell \subset \mathcal{Y} \cup M_{i_0}$, we get readily that the polynomial

$$M_{i_0} p_{A, \mathcal{Y}_\ell}^* \in \Pi_{k-2}$$

is a fundamental polynomial of A in \mathcal{X}_ℓ . We get also that it divides the polynomial $p_{A, \mathcal{X}}^* = M_{i_0} p_{A, \mathcal{Y}}^*$. \square

Below we bring some simple consequences of the fact that the set \mathcal{X}_ℓ is $(k-2)$ -independent:

Corollary 3.2. *Assume that the conditions of Proposition 3.1 hold. Then the following hold.*

- (i) $|\mathcal{X}_\ell| \leq \binom{k}{2}$;
- (ii) \mathcal{X}_ℓ contains at most $k-1$ collinear nodes;

(iii) For any curve q of degree $m \leq k - 2$ we have that

$$|\mathcal{X}_\ell \cap q| \leq d(k - 2, m).$$

Note, that ii) is a special case of iii) when $m = 1$. Let us mention that i) and ii) were proved in [9], Theorem 4.5.

3.2 On n -node lines in GC_n sets

Next, let us present a main result of this paper:

Theorem 3.3. *Assume that Conjecture 1.14 holds for all degrees up to ν . Let \mathcal{X} be a GC_n set, $n \leq \nu$ and ℓ be a line passing through exactly n nodes of the set \mathcal{X} . Then we have that*

$$|\mathcal{X}_\ell| = \binom{n}{2} \quad \text{or} \quad \binom{n-1}{2}. \quad (3.1)$$

Also, the following hold:

- (i) If $|\mathcal{X}_\ell| = \binom{n}{2}$ then there is a maximal line M_0 such that $M_0 \cap \ell \notin \mathcal{X}$. Moreover, we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup M_0)$. Hence it is a GC_{n-2} set;
- (ii) If $|\mathcal{X}_\ell| = \binom{n-1}{2}$ then there are two maximal lines M', M'' , such that $M' \cap M'' \cap \ell \in \mathcal{X}$. Moreover, we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup M' \cup M'')$. Hence it is a GC_{n-3} set.

Let us first assume that Theorem is valid and prove the following

Corollary 3.4. *Assume that the conditions of Theorem 3.3 take place. Then the following hold for any maximal line M of \mathcal{X} :*

- (i) $|M \cap \mathcal{X}_\ell| = 0$ if
 - a) $M \cap \ell \notin \mathcal{X}$ or if
 - b) there is another maximal line M' such that $M \cap M' \cap \ell \in \mathcal{X}$;
- (ii) $|M \cap \mathcal{X}_\ell| = s - 1$ if $|\mathcal{X}_\ell| = \binom{s}{2}$, where $s = n, n - 1$, for all the remaining maximal lines.

Proof of Corollary 3.4. The statements of i) concerning a) and b) follow from Lemma 1.23 and Lemma 1.24, respectively.

For the statement ii) assume that M is a maximal line intersecting ℓ at a node A and there is no other maximal line passing through that node. Now suppose that $|\mathcal{X}_\ell| = \binom{n}{2}$. Then, in view of Theorem 3.3, there is a maximal line M_0 such that $M_0 \cap \ell \notin \mathcal{X}$. According to Lemma 1.23 we have

that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup M_0)$. Therefore we get $|M \cap \mathcal{X}_\ell| = |M \cap [\mathcal{X} \setminus (\ell \cup M_0)]| = (n+1) - 2 = n-1$, since M intersects ℓ and M_0 at two distinct nodes.

Next suppose that $|\mathcal{X}_\ell| = \binom{n-1}{2}$. Then there are two maximal lines M' and M'' such that $M' \cap M'' \cap \ell \in \mathcal{X}$. Now, according to Lemma 1.24, we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup M' \cup M'')$. Therefore we get $|M \cap \mathcal{X}_\ell| = |M \cap [\mathcal{X} \setminus (\ell \cup M' \cup M'')]| = (n+1) - 3 = n-2$, since M intersects ℓ, M' and M'' at three distinct nodes. \square

Remark 3.5. Assume that the conditions of Theorem 3.3 take place and $\mathcal{X}_\ell \neq \emptyset$. Assume also that M is a maximal line of \mathcal{X} such that M intersects ℓ at a node and no node from M uses ℓ . Then there is another maximal line M' such that $M \cap M' \cap \ell \in \mathcal{X}$ and therefore no node from M' uses ℓ either.

3.3 The proof of Theorem 3.3

Let us start with

3.3.1 The case $n = 1$

GC_1 sets consist of 3 non-collinear nodes. Consider a such set $\mathcal{X} = \{A, B, C\}$ and an 1-node line ℓ that passes, say, through A . We have that no 1-node line is used in GC_n sets. Thus $\mathcal{X}_\ell = \emptyset$. Therefore we may assume that both equalities in (3.1) take place. Note also that both implications i) and ii) of Theorem 3.3 take place. Indeed, the maximal line through B and C does not intersect ℓ at a node. And at the same time the other two maximal lines, i.e., 2-node lines through A, B and A, C intersect the line ℓ at the node A .

3.3.2 The case $n = 2$

We divide this case into 2 parts.

1. GC_2 sets with 3 maximal lines:

Consider a GC_2 set \mathcal{X} with exactly 3 maximal lines. These lines intersect each other at 3 non-collinear nodes, called vertices. Except these 3 nodes, there are 3 more (non-collinear) nodes in \mathcal{X} , one in each maximal line, called "free" nodes. Here the 2-node lines are of 2 types:

a) 2-node line ℓ that does not pass through a vertex. Notice that ℓ is used only by one node and the implication i) of Theorem holds. Namely, there is a maximal line that does not intersect ℓ at a node.

b) 2-node line that passes through a vertex. Notice that no node uses a such line and the implication ii) holds.

2. GC_2 sets with 4 maximal lines:

In this case we have the Chung-Yao lattice (see Subsection 1.4). Here all 6 nodes of \mathcal{X} are intersection nodes of the maximal lines and the only used

lines are the maximal lines. Thus in this case any 2-node line is not used and evidently the implication ii) holds.

3.3.3 The case $n = 3$

We divide this case into 3 parts:

1. *The case of GC_3 sets with exactly 3 maximal lines:*

Consider a GC_3 set \mathcal{X} with exactly 3 maximal lines. By the properties of maximal lines we have that they form a triangle and the vertices are nodes of \mathcal{X} . There are 6 ($= 3 \times 2$) more nodes, called "free", 2 in each maximal line. There is also one node outside the maximal lines, denoted by O . We find readily that the 6 "free" nodes are located in 3 lines passing through O , 2 in each line (see Fig. 3.1).

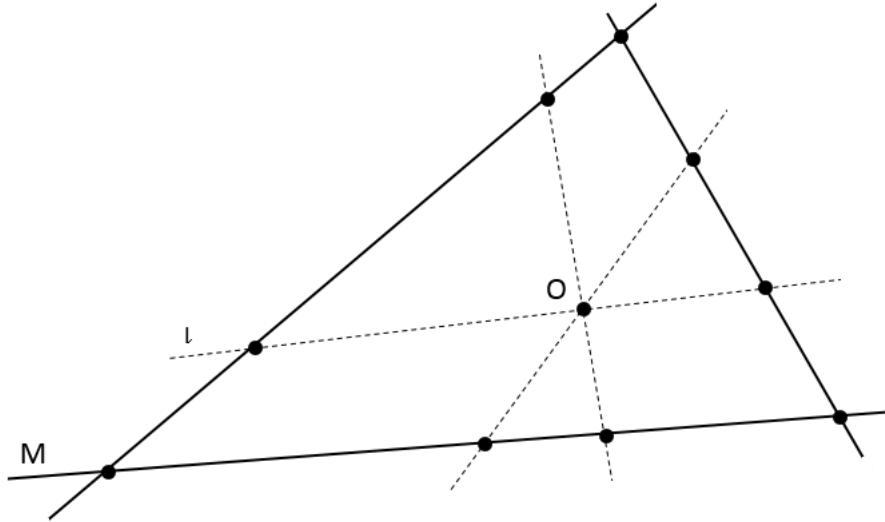


Figure 3.1: Three 3-node lines

These 3 lines are the only 3-node lines in this case. We have that for a such line ℓ there is a maximal line M that does not intersect ℓ at a node, i.e., the implication i) of Theorem holds. Also we have that ℓ is used by exactly 3 nodes. Namely, by the nodes that do not belong to $\ell \cup M$.

2. *The case of GC_3 sets with exactly 4 maximal lines:*

Now consider a GC_3 set \mathcal{X} with exactly 4 maximal lines. In this case there are 6 ($= \binom{4}{2}$) nodes that are intersection points of maximal lines. Also there are 4 more nodes in maximal lines, called "free", 1 in each. The

4 "free" nodes are not collinear.

Again we have two types of 3-node lines here.

a) 3-node line ℓ that passes through an intersection node (see Fig 3.2).

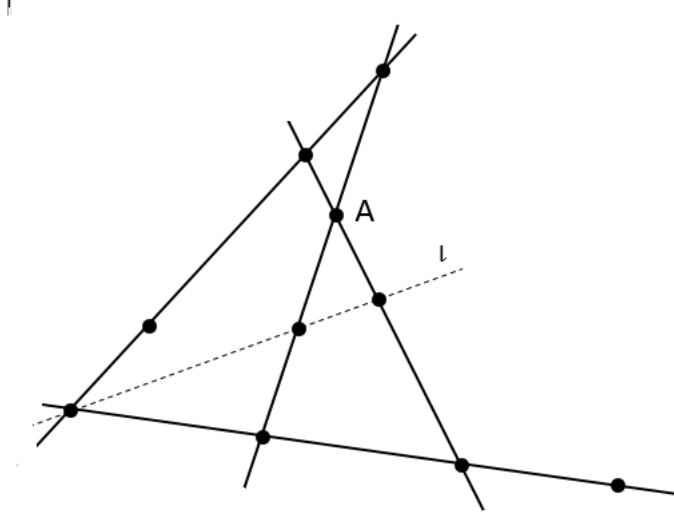


Figure 3.2: 3-node line passing through an intersection node

Note that a 3-node line can pass through at most one such node. Indeed, if a line passes through two intersection nodes then it cannot pass through any third node.

Notice that ℓ is used by only one node A and the implication ii) of Theorem takes place.

b) 3-node line ℓ that passes through 3 "free" nodes (see Fig 3.3).

Notice that the maximal line M whose "free" node is not lying in ℓ does not intersect ℓ at a node. Thus the implication i) holds. In this case ℓ is used by exactly 3 nodes. Namely, by the nodes that do not belong to $\ell \cup M$.

3. The case of GC_3 sets with exactly 5 maximal lines:

In this case we have the Chung-Yao lattice (see Subsection 1.4). Here all 10 nodes of \mathcal{X} are intersection nodes of 5 maximal lines and the only used lines are the maximal lines. Let us verify that in this case there is no 3-node line. Assume conversely that ℓ is a such line. Then through each node there pass two maximal lines and all these maximal lines are distinct. Therefore we get 6 maximal lines, which is a contradiction.

3.4 The proof of Theorem 3.3 for $n \geq 4$

We will prove Theorem by induction on n . The cases $n \leq 3$ were verified. Assume Theorem is true for all degrees less n and let us prove that it is true

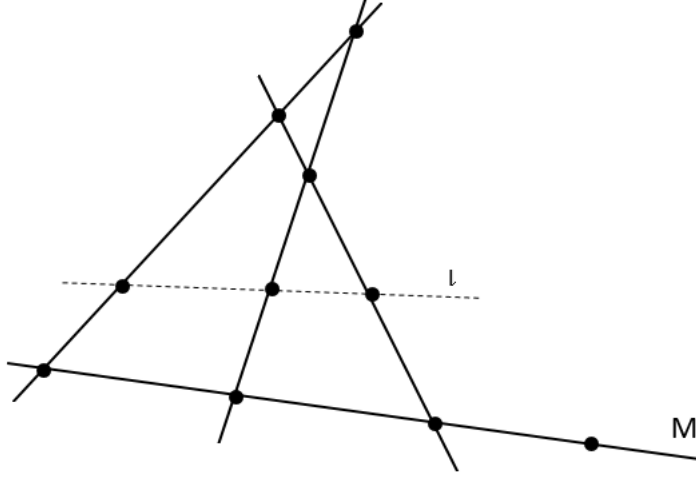


Figure 3.3: 3-node line through 3 "free" nodes

for the degree n , where $n \geq 4$.

Suppose that $|\mathcal{X}_\ell| \geq \binom{n-1}{2} + 1$. Then by assertion ii) of Proposition 2.1 we get that there is a maximal line M_0 such that $M_0 \cap \ell \notin \mathcal{X}$. Thus, in view of Lemma 1.23, we obtain that $|\mathcal{X}_\ell| = \binom{n}{2}$ and the implication i) holds.

Thus to prove Theorem it suffices to assume that

$$|\mathcal{X}_\ell| \leq \binom{n-1}{2} \quad (3.2)$$

and to prove that the implication ii) holds, i.e., there are two maximal lines M', M'' , such that $M' \cap M'' \cap \ell \in \mathcal{X}$. Indeed, this completes the proof in view of Lemma 1.24.

First suppose that two nodes in some maximal line M use the line ℓ , i.e.,

$$|M \cap \mathcal{X}_\ell| \geq 2. \quad (3.3)$$

We have that $\mathcal{X} \setminus M$ is a GC_{n-1} -set. Hence, by making use of (3.3) and induction hypothesis, we obtain that

$$|\mathcal{X}_\ell| \geq |(\mathcal{X} \setminus M)_\ell| + 2 \geq \binom{n-2}{2} + 2.$$

Therefore, in view of the condition (3.2) and Proposition 2.1 iii), we conclude that

$$|\mathcal{X}_\ell| = \binom{n-1}{2} \text{ and } \mathcal{N}_\ell \subset \beta \in \Pi_2, |\mathcal{N}_\ell| = 2n. \quad (3.4)$$

Let us use the induction hypothesis. By taking into account the first equality above and condition (3.3), we obtain that

$$|(\mathcal{X} \setminus M)_\ell| = \binom{n-2}{2}.$$

Thus the cardinality of the set $\mathcal{N}_\ell \cap (\mathcal{X} \setminus M)$ equals to $2n - 2$ ($= 2(n - 1)$). Therefore, in view of the second equality in (3.4), by using induction hypothesis we get that all the nodes in β except possibly two are located on two maximal lines of the set $\mathcal{X} \setminus M$, denoted by M' and M'' , which intersect at a node $A \in \ell$. Since $n \geq 4$ each of these two maximal lines passes through at least 3 nodes except A , which belong to β . Thus each of them divides β and we get $\beta = M'M''$. Finally, according to Proposition 2.1 iii), each of these lines passes through exactly n nodes outside ℓ and therefore they are maximal also for the set \mathcal{X} . Hence the implication ii) holds.

Thus we may suppose that

$$|M \cap \mathcal{X}_\ell| \leq 1 \quad \text{for each maximal line } M \text{ of the set } \mathcal{X}. \quad (3.5)$$

Next let us verify that we may suppose that

$$|M \cap \mathcal{X}_\ell| = 1 \quad \text{for each maximal line } M \text{ of the set } \mathcal{X}. \quad (3.6)$$

Indeed, suppose by way of contradiction that no node, say in a maximal line M_1 uses the line ℓ . Now, in view of Theorem 1.15, consider two other maximal lines of \mathcal{X} and denote them by M_i , $i = 2, 3$.

In view of the condition (3.2) and Lemma 1.23 we have that there is no maximal line M_0 such that $M_0 \cap \ell \notin \mathcal{X}$, i.e., all the maximal lines of \mathcal{X} intersect the line ℓ at a node of \mathcal{X} . Then as was mentioned above, if there are two maximal lines intersecting at a node in ℓ then Theorem follows from Lemma 1.24.

Thus, we may suppose that the 3 maximal lines M_i , $i = 1, 2, 3$, intersect the line ℓ at 3 distinct nodes, denoted by C_i , $i = 1, 2, 3$, respectively.

Then consider the GC_{n-1} -set $\mathcal{X}_2 := \mathcal{X} \setminus M_2$. We may assume that $(\mathcal{X}_2)_\ell \neq \emptyset$. Indeed, otherwise by induction hypothesis and (3.1) we would obtain that $n - 1 = 2$, i.e., $n = 3$. In \mathcal{X}_2 no node of the maximal line M_1 uses ℓ . By induction hypothesis, in view of Remark 3.5, we have that there is a maximal line M'_1 of this set intersecting ℓ at C_1 . In the same way we get that there is a maximal line M''_1 in the set $\mathcal{X} \setminus M_3$ intersecting ℓ at C_1 . Now if the maximal line M'_1 coincides with M''_1 then we get readily that it is maximal also for \mathcal{X} which completes the proof in view of Lemma 1.24. Thus suppose that the maximal lines M'_1 and M''_1 are distinct. Then consider the GC_{n-2} -set $\mathcal{X} \setminus (M_2 \cup M_3)$. Here we have 3 maximal lines M_1, M'_1 and M''_1 intersecting at the node C_1 , which is a contradiction.

Thus we have that (3.6) holds, i.e., there is only one node in each maximal line M_i , $i = 1, 2, 3$, using the line ℓ . Notice that at most one node can be

intersection node of these 3 maximal lines, since otherwise we would have 2 nodes in a maximal line that use ℓ . Consider a node A which lies, say in M_3 , uses ℓ and is not an intersection node, i.e., does not lie in the maximal lines M_1 and M_2 (see Fig. 3.4).

Consider the GC_{n-1} node set $\mathcal{X}_i := \mathcal{X} \setminus M_i$ for any fixed $i = 1, 2$. In the maximal line M_3 there is only one node using ℓ . Therefore, in view of the induction hypothesis and Corollary 3.4, we have that

$$|(\mathcal{X}_i)_\ell| = 1, \quad i = 1, 2. \quad (3.7)$$

We may conclude from here that there is only one node in $M_1 \cup M_2$, namely the intersection node $B := M_1 \cap M_2$, that uses the line ℓ .

At the same time we get from (3.7) also that $(n-1) = 2$, or $(n-1)-1 = 2$. Therefore $n \leq 4$, i.e., we may assume that $n = 4$.

3.4.1 A special case

Thus it remains to consider the case $n = 4$ with $|\mathcal{X}_\ell| = 2$. Recall that one of the nodes: A belongs to only one maximal line M_3 . While the other node: B is the intersection node of the maximal lines M_1 and M_2 (see Fig. 3.4). We will show that this case is not possible.

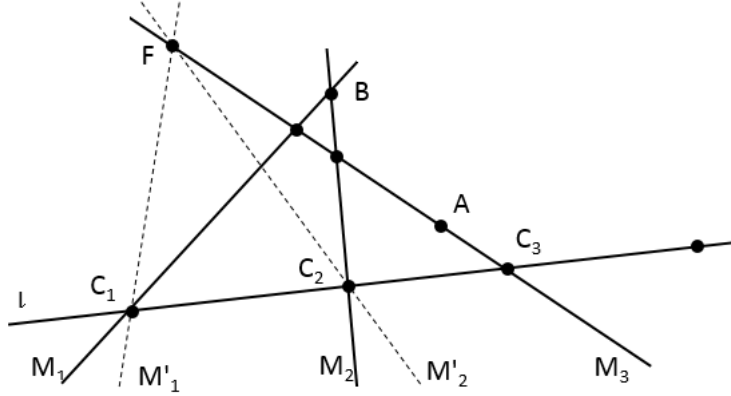


Figure 3.4: A special case

Consider the GC_3 -set $\mathcal{X}_1 := \mathcal{X} \setminus M_1$. The line ℓ is used by one node here: A and no node in maximal line M_2 uses it. Thus we conclude that there is a maximal M'_2 passing through C_2 .

Now, denote by E , the intersection node of the maximal lines M'_2 and M_3 . Let us identify this node among the 5 nodes in M_3 . Notice that evidently E is different from C_3 - the intersection node with ℓ .

We have that E is different also from the intersection nodes with M_1 or with M_2 . Indeed, three maximal lines cannot intersect at a node.

Finally note that E is different also from the node A , since it uses ℓ and therefore it does not belong to M'_2 .

Thus E coincides necessarily with the fifth node in M_3 denoted by F .

Now consider the GC_3 -set $\mathcal{X}_2 := \mathcal{X} \setminus M_2$. Again the line ℓ is used by one node here: A and no node in maximal line M_1 uses it. Thus we conclude that there is a maximal M'_1 passing through C_1 .

Then, exactly in the same way as above, we may conclude that M'_1 intersects M_3 at F .

Finally, consider the GC_2 -set $\mathcal{Y} := \mathcal{X} \setminus (M_1 \cup M_2)$. Notice that the lines M'_1, M'_2 and M_3 are 3 maximal lines intersecting at the node F , which is a contradiction. \square

Remark 3.6. Let us mention that in the cases $n \leq 5$ Theorem 3.3 is valid without the assumption concerning the Gasca-Maeztu conjecture.

3.5 A conjecture concerning GC_n sets

Conjecture 3.7. *Assume that Conjecture 1.14 holds for all degrees up to ν . Let \mathcal{X} be a GC_n set, $n \leq \nu$ and ℓ be a line passing through exactly k nodes of \mathcal{X} set. Then we have that*

$$|\mathcal{X}_\ell| = \binom{s}{2}, \text{ for some } 2k - n - 1 \leq s \leq k. \quad (3.8)$$

Moreover, for any maximal line M of \mathcal{X} we have:

- (i) $|M \cap \mathcal{X}_\ell| = 0$ if
 $M \cap \ell \notin \mathcal{X}$ or if
there is another maximal line M' such that $M \cap M' \cap \ell \in \mathcal{X}$;
- (ii) $|M \cap \mathcal{X}_\ell| = s - 1$ if $\binom{s}{2} = |\mathcal{X}_\ell|$, where $2k - n - 1 \leq s \leq k$,
for all the remaining maximal lines.

References

- [1] V. Bayramyan, On the usage of 2-node lines in n -poised sets, International Conference Harmonic analysis and approximations, 12–18 september, 2015, Tsaghkadzor, Armenia, Abstracts.

- [2] V. Bayramyan, H. Hakopian and S. Toroyan, A simple proof of the Gasca-Maeztu conjecture for $n=4$, accepted in Jaén J. Approx. **7**(1) (2015), 137–147.
- [3] V. Bayramyan, H. Hakopian and S. Toroyan, On the uniqueness of algebraic curves, Proc. of YSU, Phys. Math. Sci., **1** (2015), 3–7.
- [4] L. Berzolari, Sulla determinazione d’una curva o d’una superficie algebrica e su alcune questioni di postulazione, Ist. Lomb. Rend. (II. Ser.) **47** (1914) 556–564.
- [5] C. de Boor, Multivariate polynomial interpolation: conjectures concerning GC-sets, Numer. Algorithms **45** (2007) 113–125.
- [6] J. R. Busch, A note on Lagrange interpolation in \mathbb{R}^2 , Rev. Un. Mat. Argentina **36** (1990) 33–38.
- [7] J. M. Carnicer and M. Gasca, Planar configurations with simple Lagrange interpolation formulae, *in*: T. Lyche and L. L. Schumaker (eds.), Mathematical Methods in Curves and Surfaces: Oslo 2000, Vanderbilt University Press, Nashville, 2001, pp. 55–62.
- [8] J. M. Carnicer and M. Gasca, A conjecture on multivariate polynomial interpolation, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.), Ser. A Mat. **95** (2001) 145–153.
- [9] J. M. Carnicer and M. Gasca, On Chung and Yao’s geometric characterization for bivariate polynomial interpolation, *in*: T. Lyche, M.-L. Mazure, and L. L. Schumaker (eds.), Curve and Surface Design: Saint Malo 2002, Nashboro Press, Brentwood, 2003, pp. 21–30.
- [10] K. C. Chung and T. H. Yao, On lattices admitting unique Lagrange interpolations, SIAM J. Numer. Anal. **14** (1977) 735–743.
- [11] D. Eisenbud, M. Green and J. Harris, Ceyley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. (N.S.) **33**(3), 295–324.
- [12] M. Gasca and J. I. Maeztu, On Lagrange and Hermite interpolation in \mathbb{R}^k , Numer. Math. **39** (1982) 1–14.
- [13] H. Hakopian, On a class of Hermite interpolation problems, Adv. Comput. Math. **12** (2000) 303–309.
- [14] H. Hakopian, K. Jetter, and G. Zimmermann, Vandermonde matrices for intersection points of curves, Jaén J. Approx. **1** (2009) 67–81.
- [15] H. Hakopian, K. Jetter, and G. Zimmermann, A new proof of the Gasca-Maeztu conjecture for $n = 4$, J. Approx. Theory **159** (2009) 224–242.

- [16] H. Hakopian, K. Jetter and G. Zimmermann, The Gasca-Maeztu conjecture for $n = 5$, *Numer. Math.* **127** (2014) 685–713.
- [17] H. Hakopian and A. Malinyan, Characterization of n -independent sets of $\leq 3n$ points, *Jaén J. Approx.* **4** (2012) 119–134.
- [18] H. Hakopian and L. Rafayelyan, On a generalization of Gasca-Maeztu conjecture, *New York J. Math.* **21** (2015) 351–367.
- [19] H. Hakopian and S. Toroyan, On the minimal number of nodes determining uniquely algebraic curves, *Proc. of YSU, Phys. Math. Sci.*, **3** (2015) 17–22.
- [20] H. Hakopian, S. Toroyan, On the uniqueness of algebraic curves passing through n -independent nodes, *arXiv:1510.05211v1[math.NA]*.
- [21] K. Jetter, Some contributions to bivariate interpolation and cubature, *in: C. K. Chui, L. L. Schumaker and J. D. Ward (eds), Approximation Theory IV, Acad. Press, New York, 1983, pp. 533–538.*
- [22] J. Radon, Zur mechanischen Kubatur, *Monatsh. Math.* **52** (1948) 286–300.
- [23] L. Rafayelyan, Poised nodes set constructions on algebraic curves, *East J. Approx.* **17** (2011) 285–298.
- [24] F. Severi, *Vorlesungen über Algebraische Geometrie*, Teubner, Berlin, 1921 (Translation into German - E. Löffler).
- [25] S. Toroyan, On a conjecture in bivariate interpolation, accepted in *Proc. of YSU*.

Vahagn Bayramyan, Hakop Hakopian

Department of Informatics and Applied Mathematics

Yerevan State University

A. Manukyan St. 1

0025 Yerevan, Armenia

E-mails: vahagn.bayramyan@gmail.com, hakop@ysu.am